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A Seventh-order Block Integrator for Solving Stiff Systems

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Abstract

In this paper, an L_0- stable Second Derivative Block Integrator of uniform order seven is proposed for the numerical integration of stiff systems, including large stiff systems resulting from semi-discretization of Parabolic differential equations. The conventional 3-step second derivative backward differentiation formula is obtained from a continuous scheme while the additional methods are obtained from the second derivative of the same continuous scheme. All methods are derived via Interpolation and Collocation techniques and assembled into a block scheme. The convergence and stability properties of the block scheme are discussed and the stability region shown. The performance of the scheme as compared to other existing schemes is considered favorable.

1. Introduction

Consider a system of first order differential equation of the form

$$u'(t) = f(t, u(t)),\ u(t_0) = u_0,\ t \in [t_0, t_N] \tag{1}$$

where $N \in \mathbb{Z}^+, f : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m, m$ is the dimension of the system and f satisfies the Lipschitz condition as given in Henrici [1]. The system (1) is said to be stiff if the eigenvalues of the Jacobian $(\frac{\partial f}{\partial y})$ matrix have negative real parts. Recently, scholars have been attracted to the research of stiff problems especially large stiff systems resulting from the semi-discretization of Parabolic equations

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since solution to most partial differential equations are not easily obtained in the closed form and numerical methods to handle such problem must have large stability intervals. Hence, the need for at least L_0- stable methods for efficiently solving (1) with large system. Stiff problems were first researched into by Curtis and Hirschfelder [2] and after then, several other scholars have proposed numerical methods efficient for solving (1) (see Enright [3], Hairer and Wanner [4], Cash [5], Chartier [6], Brugnano and Trigiante [7], Onumanyi et al [9]).
In this paper, the conventional 3-step second derivative backward differentiation formula (main method) and additional methods are assembled as a block scheme (Block Second-derivative Backward Differentiation Formula denoted as BSBDF) to solve large stiff systems. The BSBDFs are bundle as main and additional methods, a concept that is due to Brugnano and Trigiante [8]. The main method is obtained from a continuous scheme while the additional methods are obtained from the second derivative of the continuous scheme. The numerical solution of the problem is simultaneously provided in each block without the use of predictor from other methods (see Jator and Agyinyi [10], Jator [11]).
The paper is structured as follows: Section 2 shows the derivation of 3-step BSBDF and how to generate the specific members of the block scheme. In section 3, the stability and convergence properties of the block scheme is studied while in section 4, the numerical algorithm of the block scheme is implemented on selected stiff systems and Parabolic equations. Finally, the conclusion of the paper is discussed in section 5.

2. Derivative of 3-step BSBDF

In this section, we construct the main method and the additional methods from its second derivative and are combined to form the 3-step Block Second derivative Backward Differentiation Formula (BSBDF) on the interval $[t_n, t_{(n+3)}]$. The conventional 3-step second derivative backward differentiation formula which is the main method is of the form

$$u_{(n+3)} = \sum_{j=0}^{2} \alpha_j u_{n+j} + h \sum_{j=0}^{3} \beta_j f_{n+j} + h^2 \gamma_3 g_{n+3} \tag{2}$$

where

$$u_{n+j} = u(t_n + jh),\ \ f_{n+j} = f(t_n + jh, u(t_n + jh)) = u'(t_n + jh)$$

and

$$g_{n+3} = \frac{df(t_n + 3h,\ u(t_n + 3h))}{dt} = u''(t_n + 3h),$$

t_n is a node point and $\alpha_j, \beta_j, j = 0, 1, ..., 3,\ \gamma_3$ are parameters to be obtained from the collocation and interpolation techniques. The exact solution $U(t)$ is assumed to exist and unique in $[t_0, t_3]$. We approximate the exact solution $U(t)$ by seeking the continuous solution $u(t)$ of the form

$$u(t) = \sum_{j=0}^{r+s-1} b_j \varphi_j(t),\ t \in [t_0, t_3] \tag{3}$$

where b_j are unknown coefficients and $\varphi_j(t) = t^j,\ j = 0(1)r+s-1$ are the polynomial basis functions of degree $r+s-1$. The number of interpolation points r and the number of the distinct collocation points s are chosen to satisfy $r = k$ and $s = k+2$. The 3-step Block Second Derivative Backward Differentiation Formula of order 7 is constructed by imposing the following conditions:

$$u_{n+i} = \sum_{j=0}^{7} b_j t_{n+i}^{j},\ i = 0(1)2 \tag{4}$$

$$f_{n+i} = \sum_{j=0}^{7} j b_j t_{n+i}^{j-1},\ i = 0(1)3 \tag{5}$$

$$g_{n+i} = \sum_{j=0}^{7} j(j-1) b_j t_{n+i}^{j-2},\ i = 3 \tag{6}$$

Equations (4)-(6) lead to a system of eight equations which is solved to obtain the coefficients $b_j, j = 0(1)7$ and substitute the values of b_j into equation (3) to obtain the continuous solution in the form of (7) as

$$u(t) = \sum_{j=0}^{2} \alpha_j(t) u_{n+j} + h \sum_{j=0}^{3} \beta_j(t) f_{n+j} + h^2 \gamma_3(t) g_{n+3} \tag{7}$$

where h is the chosen step-length and $\alpha_j(t), j = 0(1)2, \beta_j(t), j = 0(1)3, \gamma_3$ are continuous coefficients given as

$$\alpha_0(t) = 1 - \frac{3051}{388}\left(\frac{t-t_n}{h}\right)^2 + \frac{5793}{388}\left(\frac{t-t_n}{h}\right)^3 - \frac{2355}{194}\left(\frac{t-t_n}{h}\right)^4 + \frac{975}{194}\left(\frac{t-t_n}{h}\right)^5 - \frac{403}{388}\left(\frac{t-t_n}{h}\right)^6 + \frac{33}{388}\left(\frac{t-t_n}{h}\right)^7$$

$$\alpha_1(t) = \frac{108}{97}\left(\frac{t-t_n}{h}\right)^2 + \frac{516}{97}\left(\frac{t-t_n}{h}\right)^3 - \frac{1005}{97}\left(\frac{t-t_n}{h}\right)^4 + \frac{628}{97}\left(\frac{t-t_n}{h}\right)^5 - \frac{166}{97}\left(\frac{t-t_n}{h}\right)^6 + \frac{16}{97}\left(\frac{t-t_n}{h}\right)^7$$

$$\alpha_2(t) = \frac{27}{4}\left(\frac{t-t_n}{h}\right)^2 + \frac{81}{4}\left(\frac{t-t_n}{h}\right)^3 - \frac{45}{4}\left(\frac{t-t_n}{h}\right)^4 + \frac{23}{4}\left(\frac{t-t_n}{h}\right)^5 - \frac{11}{4}\left(\frac{t-t_n}{h}\right)^6 + \frac{1}{4}\left(\frac{t-t_n}{h}\right)^7$$

$$\beta_0(t)=\left(\frac{t-t_n}{h}\right)-\frac{1051}{291}\left(\frac{t-t_n}{h}\right)^2+\frac{54259}{10476}\left(\frac{t-t_n}{h}\right)^3-\frac{3292}{873}\left(\frac{t-t_n}{h}\right)^4+\frac{7721}{5238}\left(\frac{t-t_n}{h}\right)^5-\frac{257}{873}\left(\frac{t-t_n}{h}\right)$$
$$+\tfrac{247}{10476}\left(\tfrac{t-t_n}{h}\right)^7$$

$$\beta_1(t)=-\frac{108}{97}\left(\frac{t-t_n}{h}\right)^2+\frac{2090}{97}\left(\frac{t-t_n}{h}\right)^3-\frac{8209}{388}\left(\frac{t-t_n}{h}\right)^4+\frac{3841}{388}\left(\frac{t-t_n}{h}\right)^5-\frac{861}{338}\left(\frac{t-t_n}{h}\right)^6$$
$$+\tfrac{75}{388}\left(\tfrac{t-t_n}{h}\right)^7$$

$$\beta_2(t)=-\frac{585}{194}\left(\frac{t-t_n}{h}\right)^2+\frac{3625}{388}\left(\frac{t-t_n}{h}\right)^3-\frac{1049}{97}\left(\frac{t-t_n}{h}\right)^4+\frac{1125}{194}\left(\frac{t-t_n}{h}\right)^5-\frac{281}{194}\left(\frac{t-t_n}{h}\right)^6$$
$$+\tfrac{53}{388}\left(\tfrac{t-t_n}{h}\right)^7$$

$$\beta_3(t)=-\frac{79}{291}\left(\frac{t-t_n}{h}\right)^2+\frac{2326}{2619}\left(\frac{t-t_n}{h}\right)^3-\frac{3853}{3492}\left(\frac{t-t_n}{h}\right)^4+\frac{6775}{1047}\left(\frac{t-t_n}{h}\right)^5-\frac{623}{2326}\left(\frac{t-t_n}{h}\right)^6$$
$$+\tfrac{193}{10476}\left(\tfrac{t-t_n}{h}\right)^7$$

$$\gamma_3(t)=-\frac{8}{97}\left(\frac{t-t_n}{h}\right)^2+\frac{239}{837}\left(\frac{t-t_n}{h}\right)^3-\frac{100}{197}\left(\frac{t-t_n}{h}\right)^4+\frac{359}{1746}\left(\frac{t-t_n}{h}\right)^5-\frac{17}{291}\left(\frac{t-t_n}{h}\right)^6$$
$$+\tfrac{11}{1746}\left(\tfrac{t-t_n}{h}\right)^7$$

The main method is generated by evaluating (7) at the point $t=t_{n+3}$ to obtain

$$(8)\quad u_{n+3}=\frac{16}{97}u_n+\frac{81}{97}u_{n+1}+\frac{h}{97}(4f_n+54f_{n+1}+108f_{n+2}+44f_{n+3})-\frac{6}{97}h^2g_{n+3}$$

Differentiating (7) twice with respect to t, we have

$$(9)\qquad u''(t)=\frac{1}{h^2}\left(\sum_{j=0}^{2}\alpha_j(t)u_{n+j}+\sum_{j=0}^{3}\beta_j(t)f_{n+j}+h^2\gamma_3(t)g_{n+3}\right)$$

where $\alpha_j(t)=\dfrac{d^2\alpha_j(t)}{dt^2}|_{j=0,1,2}\ ,\ \beta_j(t)=\dfrac{d^2\beta_j(t)}{dt^2}|_{j=0,1,2,3}\ ,\ \gamma_j(t)=\dfrac{d^2\gamma_j(t)}{dt^2}$

. The additional method is generated by evaluating (9) at the point $t=t_{n+1},t_{n+2}$ to obtain

(10)
$$h^2 g_{n+1} = \frac{2916}{2619} u_n - \frac{13392}{2619} u_{n+1} + \frac{10476}{2619} u_{n+2} + \frac{h}{2619} \left(632 f_n - 4563 f_{n+1} - 3888 f_{n+2} + 259 f_{n+3}\right) - \frac{75}{2619} h^2 g_{n+3}$$

(11)
$$h^2 g_{n+2} = \frac{3321}{5238} u_n + \frac{25488}{5238} u_{n+1} - \frac{28809}{5238} u_{n+2} + \frac{h}{5238} \left(806 f_n + 13500 f_{n+1} + 16524 f_{n+2} + 1300 f_{n+3}\right) - \frac{336}{5238} h^2 g_{n+3}$$

The methods (8), (10) and (11) are combined as a 3-step BSBDF of order 7 given by

$$h^2 g_{n+1} = \frac{2916}{2619} u_n - \frac{13392}{2619} u_{n+1} + \frac{10476}{2619} u_{n+2} + \frac{h}{2619} \left(632 f_n - 4563 f_{n+1} - 3888 f_{n+2} + 259 f_{n+3}\right) - \frac{75}{2619} h^2 g_{n+3)}$$

$$h^2 g_{n+2} = \frac{3321}{5238} u_n + \frac{25488}{5238} u_{n+1} - \frac{28809}{5238} u_{n+2} + \frac{h}{5238} \left(806 f_n + 13500 f_{n+1} + 16524 f_{n+2} + 1300 f_{n+3}\right) - \frac{336}{5238} h^2 g_{n+3}$$

(12)
$$u_{n+3} = \frac{16}{97} u_n + \frac{81}{97} u_{n+1} + \frac{h}{97} \left(4 f_n + 54 f_{n+1} + 108 f_{n+2} + 44 f_{n+3}\right) - \frac{6}{97} h^2 g_{n+3}$$

3. Analysis of 3-step BSBDF

The 3-step BSBDF (12) is represented by a block matrix finite difference equation as

$$M_1 U_\omega = M_0 U_{\omega-1} + h \left(N_1 F_\omega + N_0 F_{\omega-1}\right) + h^2 P_1 G_\omega \quad (13)$$

where

$$U_\omega = \left(u_{n+1}, u_{n+2},\ u_{n+3}, ..., u_{n+k-1}, u_{n+k}\right)^T$$

$$U_{\omega-1} = \left(u_{n-k+1}, u_{n-k+2}, u_{n-k+3} ..., u_{n-1}, u_n\right)^T$$

$$F_\omega = \left(f_{n+1}, f_{n+2}, f_{n+3}, ..., f_{n+k-1}, f_{n+k}\right)^T$$

$$F_{\omega-1} = \left(f_(n-k+1, f_{n-k+2}, f_{n-k+3}, ..., f_{n-1}, f_n\right)^T$$

$$G_\omega = \left(g_{n+1}, g_{n+2}, g_{n+3}, ..., g_{n+k-1}, g_{n+k}\right)^T$$

and

$$M_1 = \begin{pmatrix} \frac{496}{97} & -4 & 0 \\ -\frac{472}{97} & \frac{11}{2} & 0 \\ -\frac{81}{97} & 0 & 1 \end{pmatrix} \quad M_0 = \begin{pmatrix} 0 & 0 & \frac{108}{97} \\ 0 & 0 & \frac{123}{194} \\ 0 & 0 & \frac{16}{97} \end{pmatrix} \quad N_1 = \begin{pmatrix} -\frac{169}{97} & -\frac{144}{97} & \frac{259}{2619} \\ \frac{250}{97} & \frac{306}{97} & \frac{650}{2619} \\ \frac{54}{97} & \frac{108}{97} & \frac{44}{97} \end{pmatrix}$$

$$N_0 = \begin{pmatrix} 0 & 0 & \frac{632}{2619} \\ 0 & 0 & \frac{403}{2619} \\ 0 & 0 & \frac{4}{97} \end{pmatrix} \quad P_1 = \begin{pmatrix} -1 & 0 & -\frac{25}{873} \\ 0 & -1 & -\frac{56}{873} \\ 0 & 0 & -\frac{6}{97} \end{pmatrix}$$

$$L(u(t_n), h) = \sum_{j=0}^{3} \big([\alpha_j u(t_n + jh) - h\beta_j u'(t_n + jh)\big) - h^2\gamma_3 u''(t_n + 3h), \tag{14}$$

Assume that $u(t)$ is differentiable as often as needed and evaluated at $t = t_n$, then by using Taylor series expansion to expand $u(t_n+jh), u'(t_n+jh)$ and $u''(t_n+3h)$ in (14) about t_n, we have

$$u(t_n + jh) = \sum_{m=0}^{\infty} \frac{(jh)^m}{m} u^m t_n,$$

$$u'(t_n + jh) = \sum_{m=0}^{\infty} \frac{(jh)^m}{m} u^{m+1} t_n,$$

$$u''(t_n + 3h) = \sum_{m=0}^{\infty} \frac{(3h)^m}{m} u^{m+2} t_n$$

Substitute $u(t_n + jh), u'(t_n + jh)$ and $u''(t_n + 3h)$ into (14) to obtain

$$L(u(t_n), h) = C_0 u(t_n) + C_1 h u'(t_n) + C_2 h^2 u'' t_n + ... + C_m h^m u^m t_n + ... \tag{15}$$

where $C_m, m = 0, 1, 2...$ are constants given in terms of α_j and β_j

$$\left.\begin{array}{c} C_0 = \sum_{j=0}^{3} \alpha_j \\ C_1 = \sum_{j=0}^{3} (j\alpha_j - \beta_j) \\ C_2 = \frac{1}{2}\sum_{j=0}^{3} j^2\alpha_j - \sum_{j=0}^{3} j\beta_j - \sum_{j=1}^{3}\gamma_j \\ C_m = \frac{1}{m!}\left(\sum_{j=0}^{3} j^m\alpha_j - m\sum_{j=0}^{3} j^{m-1}\beta_j - m(m-1)\sum_{j=1}^{3} j^{m-2}\gamma_j\right) \end{array}\right\} \tag{16}$$

The BSBDF method (12) is said to have a maximal order of accuracy m if

$$C_0 = C_1 = C_2 = ... = C_m = 0, \; C_(m + 1) \neq 0$$

Thus the 3-step BSBDF is of order 7 since

$$C_0 = C_1 = C_2 = ... = C_7 = 0, \; C_8 \neq 0$$

with error constant

$$C_8 = \left(\frac{61}{244440}, \frac{17}{54320}, \frac{3}{27160}\right)^T$$

3.1. **Zero-stability of BSBDF.** Applying 3-step BSBDF to the scalar test problem $u' = \lambda u$ and let $z = h\lambda$, we have

$$M_1 - zN_1 - z^2P_1 = \begin{pmatrix} \frac{496}{97} + \frac{169}{97}z + z^2 & -4 + \frac{144}{97}z & -\frac{259}{2619}z + \frac{25}{873}z^2 \\ -\frac{472}{97} - \frac{250}{97}z & \frac{11}{2} - \frac{306}{97}z + z^2 & -\frac{650}{2619}z + \frac{56}{873}z^2 \\ -\frac{81}{97} - \frac{54}{97}z & -\frac{108}{97}z & 1 - \frac{44}{97}z + \frac{6}{97}z^2 \end{pmatrix}$$

$$M_0 + zN_0 = \begin{pmatrix} 0 & 0 & \frac{108}{97} + \frac{632}{2619}z \\ 0 & 0 & \frac{123}{194} + \frac{403}{2619}z \\ 0 & 0 & \frac{16}{97} + \frac{4}{97}z \end{pmatrix}$$

Since $R(\mu, z) = det\left((M_1 - zN_1 - z^2P_1)\mu - (M_0 + zN_0)\right)$. The stability polynomial $R(\mu, z)$ of 3-step BSBDF is

$$\begin{aligned} R(\mu, z) = {} & \frac{840\mu^2}{97} - \frac{840\mu^3}{97} + \frac{1080\mu^2 z}{97} + \frac{1440\mu^3 z}{97} + \frac{620\mu^2 z^2}{97} - \frac{1160\mu^3 z^2}{97} \\ & + \frac{204\mu^2 z^3}{97} + \frac{576\mu^3 z^3}{97} + \frac{40\mu^2 z^4}{97} - \frac{193\mu^3 z^4}{97} + \frac{4\mu^2 z^5}{97} + \frac{44\mu^3 z^5}{97} - \frac{6\mu^3 z^6}{23} \end{aligned} \tag{17}$$

Solving (17) as $z \longrightarrow 0$, the first characteristics polynomial $\rho(\mu)$ is given as

$$\rho(\mu) = \frac{840}{97}\mu^2(1 - \mu)$$

Following Fatunla [12], the 3-step BSBDF is zero-stable since the roots of $\rho(\mu) = 0$ are less than or equal to 1 and for the root equal to 1, the multiplicity does not exceed 1.

3.2. **Consistency and Convergence.** Since 3-step BSBDF is of order $p = 7 > 1$, therefore the method is consistent. The necessary and sufficient condition for convergence is that the method is consistent and zero-stable, therefore the 3-step BSBDF is convergent.

3.3. **Region of Absolute Stability.** From (17), equate $R(\mu, z) = 0$ and solve for μ gives the roots of the stability polynomial as $\mu_1 \longrightarrow 0, \mu_2 \longrightarrow 0,$

$$\mu_3 \longrightarrow \frac{8.65979 + 11.134z + 6.39175z^2 + 2.10309z^3 + 0.412371z^4 + 0.0412371z^5)}{8.65979 - 14.8454z + 11.9588z^2 - 5.93814z^3 + 1.98969z^4 - 0.453608z^5 + 0.0618557z^6}$$

The stability function of the method is the maximum root of the three roots which is

$$\mu_3(z) = \frac{8.65979 + 11.134z + 6.39175z^2 + 2.10309z^3 + 0.412371z^4 + 0.0412371z^5}{8.65979 - 14.8454z + 11.9588z^2 - 5.93814z^3 + 1.98969z^4 - 0.453608z^5 + 0.0618557z^6} \tag{18}$$

Set $|\mu_3(z)| \leq 1$ to obtain the absolute stability interval which is shown in the region of absolute stability. The RAS of 3-step BSBDF (12) is plotted in fig.1.

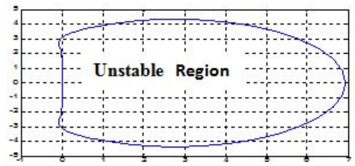

Fig. 1: Region of Absolute Stability of 3-step BSBDF

From Fig. 1, since the method is A_0-stable and in addition,

$$\lim_{z \longrightarrow \infty} \mu_3(z) = 0$$

Thus 3-step BSBDF is L_0-stable.

4. Numerical Examples

This section deals with some numerical examples executed in MAPLE language to show the performance of the Block method on selected stiff problems.

Example 1:

Consider the following linear system on the range $0 \leq t \leq 1$

$$u' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} u, \; u(0) = ((1 \; 0 \; -1))^T$$

with theoretical solution

$$u_1(t) = \frac{1}{2}e^{-2t} + e^{-40t}\left(cos(40t) + sin(40t)\right)$$

$$u_2(t) = \frac{1}{2}e^{-2t} - e^{-40t}\left(cos(40t) + sin(40t)\right)$$

$$u_3(t) = \frac{1}{2}2e^{-40t}\left(sin(40t) - cos(40t)\right)$$

This problem has been solved by Brugnano and Trigiante [7] using the GBDF (Generalized Backward Differentiation Formula) of order 7 and also solved by Akinfenwa et al. [13] using BBDF (Block Backward Differentiation Formula) of order 7. The results for the GBDF and BBDF are reproduced in Table 1 and compared with the results given by BSBDF of order 7. It is seen from Table 1 that the BSBDF performs better than both GBDF and BBDF.

Example 2:
Consider the linear system on the range $0 \le t \le T$

$$u_1' = -10u_1 + \beta u_2,\ u_1(0) = 1$$
$$u_2' = -\beta u_1 - 10u_2,\ u_2(0) = 1$$
$$u_3' = -\gamma u_3,\ u_3(0) = 1$$

This problem was extensively solved by Shampine [15] and reported that the system is stiff when $\beta = 21$ and $\gamma = 10$ and the Jacobian has eigenvalues $-10 \pm \beta i$ and γ. Its exact solution is:

$$u_1(t) = e^{-\gamma t}\left(cos(\beta t) + sin(\beta t)\right)$$
$$u_2(t) = e^{-\gamma t}\left(cos(\beta t) - sin(\beta t)\right)$$
$$u_3(t) = e^{-\gamma t}$$

The absolute error for the end point $T = 1$ and 2 are computed and shown in Tables 2 and 3 respectively. The problem was also solved by Akinfenwa et al.[14] using Continuous Block Hybrid Method (CBHM) of order 9. The results for the CBHM are reproduced in Tables and compared with the results given by BSBDF of order 7. From Tables 2 and 3, it is seen that the BSBDF performs better than existing method CBHM.

Example 3:
Consider the nonlinear system which was solved by Jator and Agyinyi [10] and also by Akinfenwa et al. [13]

$$u_1' = -1002u_1 + 1000u_2,\ u_1(0) = 1$$
$$u_2' = u_1 - u_2(1 + u_2),\ u_2(0) = 1$$

Exact solution $u_1(t) = e^{-2t},\ u_2(t) = e^{-t}$
It is obvious from Table 4 that the absolute errors obtained at the specified values of t for BBDF and BSBDF are small and BSBDF performs better than BBDF. We chose not to compare BSBDF with HBDF in Jator and Agyinyi [10] since BSBDF is expected to perform better because of its higher order.

Example 4:
Consider the Parabolic Partial Differential Equation of the form, (see Cash [16]),

$$\frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial x^2} = 0,\ 0 < x < 1,\ 0 < t$$
$$u(0, t) = u(1, t) = 0$$
$$u(x, 0) = sin(\pi x) + sin(w\pi x),\ 0 \le x \le 1,\ w >> 1$$

The exact solution to this problem is

$$u(x, t) = e^{-(\pi^2 \mu t)} sin(\pi x) + e^{-(w^2 \pi^2 \mu t)} \sin w(\pi x)$$

Semi-discretizing the PDE to a system of IVP using Method of lines (Lambert [17], Ramos and Vigo-Aguiar [18]),

$$\frac{\partial u}{\partial t} = \frac{\partial u_i}{\partial t}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\}$$

. Thus the PDE transforms to a system of ODEs

$$\frac{du}{dx} = \frac{\mu}{h^2}\{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\}\ i = 1, 2, , N-1, t > 0$$

Also written as

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ . \\ . \\ u_{N-1}(t) \\ u_{N-2}(t) \end{pmatrix} = \frac{\mu}{h^2} \begin{pmatrix} -2 & 1 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & ... & ... \\ 0 & 0 & ... & ... & ... & ... \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & ... & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ . \\ . \\ u_{N-1}(t) \\ u_{N-2}(t) \end{pmatrix}, t > 0$$

with initial conditions

$$u(x_i, 0) = \sin(\pi x_i) + \sin(w\pi x_i),\ 0 \leq x_i \leq 1\ , i = 1, 2, ..., N-1$$

BSBDF are applied to the system of IVP arising from the parabolic equation and the results are compared to existing method also used in solving the problem in Cash [16] and Crank-Nicholson (C-N). Table 5 shows the absolute error of the methods at $t = 1, \mu = 1$ at different w values. Cash [16] notes that as w increases, the problem in example 4 exhibits features similar to stiff equations and that methods like Crank-Nicolson which are not L_0-stable are expected to perform poorly. The BSBDF is L_0-stable and hence perform well and better than the existing method in Cash [16].

Table 1: A comparison of Errors of Methods for Example 1

H	BBDF(order 7) Max $\|error\|$	GBDF(order 7) Max $\|error\|$	BSBDF(order 7) Max $\|error\|$
0.01	1.81E-4	1.18E-3	1.13E-6
0.005	3.37E-6	1.38E-5	1.31E-9
0.0025	2.03E-8	1.07E-7	1.43E-11
0.00125	1.57E-10	1.07E-9	1.41E-13
0.000625	1.16E-12	9.41E12	1.23E-15

Table 2: A Comparison of errors of Methods for Example 2 at $T = 1$ where Error=$|u_i - u_i(t_i)|$

	BBDF(order 9)	GBDF(order 7)	BSBDF(order 7)			
h	Error 1	Error 2	Error 3	Error 1	Error 2	Error 3
0.01	3.46E-6	2.43E-5	1.68E-6	3.17E-6	1.13E-5	3.15E-8
0.01	1.26E-6	1.12E-5	1.01E-7	2.45E-12	5.46E-12	5.03E-15
0.001	1.74E-6	1.09E-5	7.46E-8	1.51E-19	6.41E-19	2.40E-22

Table 3: A Comparison of errors of Methods for Example 2 at $T = 2$

	BBDF(order 9)	GBDF(order 7)	BSBDF(order 7)			
h	Error 1	Error 2	Error 3	Error 1	Error 2	Error 3
0.01	1.74E-9	1.11E-9	1.97E-10	5.74E-10	7.04E-10	2.73E-12
0.01	8.36E-10	3.84E-10	9.18E-12	2.89E-16	4.52E-16	4.51E-19
0.001	8.54E-10	3.51E-10	6.78E-12	4.12E-23	4.33E-23	2.03E-26

Table 4: A comparison of absolute errors of Methods for Example 3

t	h	N	BBDF(order 7)		BSBDF(order 7)	
			Err(u_1)	Err(u_2)	Err(u_1)	Err(u_2)
1	0.05	20	8.4994E-12	1.2500E-11	2.9131E-14	3.9452E-14
10	0.017	600	6.5265E-22	7.1828E-18	1.7528E-24	1.9179E-20

Table 5: Absolute Errors of methods for Example 4 at $t = 1, \mu = 1$

w	BSBDF	C-N	Cash [16]
1	2.69E-06	6.20E-05	3.7E-05
2	1.35E-06	3.83E-05	1.8E-05
3	1.35E-06	9.30E-03	1.9E-05
5	1.35E-06	1.80E-01	1.8E-05
10	1.35E-06	6.10E-01	1.8E-05

5. Conclusion

An L_0- stable Block Second-derivative Backward Differentiation Formula (BS-BDF) has been proposed and implemented as a self-starting method for the numerical integration of stiff systems which includes large systems arising from semi-discretized parabolic PDEs. The convergence, consistency and stability of the block method were established. Application of the block algorithm to linear problems, non-linear problem and a system of initial value problem resulting from discretizing parabolic equation demonstrate the accuracy of the method. BSBDF is clearly superior when compared to some existing schemes of same or higher order in literature.

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