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L-Stable Block Backward Differentiation Formula for Parabolic Partial Differential Equations



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Abstract In this paper, an L-stable Second Derivative Block Backward Differentiation Formula (SDBBDF) of order 5 is presented for the solutions of parabolic equations. It applied the use of the classical method of lines for the discretization of the parabolic equations. The method reduces the one-dimensional parabolic partial differential equation which has integral or non-integral boundary conditions to a system of Ordinary Differential Equations (ODEs) with initial conditions. The stability properties of the block method are investigated using the boundary locus plot and the method was found to be L-stable. The derived method is implemented on standard problems of parabolic equations and the results obtained show that the method is accurate and efficient.

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1. Introduction

Consider the second-order Parabolic Differential Equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + Q(x, t), \quad 0 < x < X, \quad 0 < t \leq T \quad (1)$$

subject to initial conditions

$$u(x, 0) = f(x), \quad 0 \leq x \leq X, \quad (2)$$

and boundary conditions

$$u(0, t) = \int_0^X r_1(x, t)u(x, t)dx + s_1(t), \quad 0 < t \leq T \quad (3)$$

$$u(X, t) = \int_0^X r_2(x, t)u(x, t)dx + s_2(t), \quad 0 < t \leq T \quad (4)$$

Or

$$u(0, t) = u(X, t) = \beta \quad t \geq 0 \quad (5)$$

where $Q(x, t), f(x), r_1(x, t), r_2(x, t), s_1(t), s_2(t)$ are given continuous functions which satisfies the existence and uniqueness conditions (see Lambert [6]). In recent years, the investigation of problems for partial differential equations with integral conditions has become very important due to their practical interpretations. Cannon [7] is one of the first researchers to investigate this class of problems where integral conditions are used for the one-dimensional heat conduction equation. Other authors that have proffer numerical methods for the solution of such problems are Li et al. [16], Dehghan [9,10] and Friedman [1]. Also, various problems arising from heat

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conduction [3], thermo-elasticity [13], plasma physics [2] and chemical engineering [17] can be reduced to integral problems. In this work, the approximate solution to the parabolic problem (1) will be given based on L-stable SDBBDF. The method of lines [15] is used to reduce (1) with its initial conditions and boundary conditions to a system of N first-ordinary differential equations with initial conditions of the form

$$\frac{dU_i}{dt} = AU_i(t) + v_i(t), \quad t > 0 \tag{6a}$$

with initial conditions

$$U_i(x, 0) = f_i(x), \quad i = 1, 2, \dots, N - 1 \tag{6b}$$

where

$$A = \frac{1}{(\Delta x)^2} \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}$$

The L-stable Second Derivative Block Backward Differentiation Formula (SDBBDF) of order 5 is developed to solve the resulting system of initial value problems (IVPs) from the parabolic equation. Block methods were first introduced by Milne [14] and since then various block methods have been developed (see [4,11,12]). Block methods preserve the Runge–Kutta traditional advantage of being self starting and also have the novel property of simultaneously producing approximate solutions of the IVPs at more than one point.

The rest of this paper is presented as follows: the numerical method is derived in Section 2. Analysis and implementation of the derived method are presented in Sections 3 and 4 respectively. The numerical results produced by this method are given in Section 5 and finally a brief conclusion is given in Section 6.

2. Derivation of the method

In this section, we develop the main method SDBBDF with its additional method derived from its second derivative combined to form the 2-step Second Derivative Block Method on the interval t_n to $t_{n+2} = t_n + 2h$ where h is the chosen step length. The main method SDBBDF is of the form

$$u_{n+2} = \sum_{j=0}^1 \alpha_j u_{n+j} + \sum_{j=0}^2 \beta_j f_{n+j} + h^2 \gamma_2 g_{n+2} \tag{7}$$

where

$$u_{n+j} = u(t_n + jh), \quad f_{n+j} \equiv f(t_n + jh, u(t_n + jh)) = u'(t_n + jh) \text{ and}$$

$$g_{n+2} \equiv \frac{df(t_n + 2h, u(t_n + 2h))}{dt} = u''(t_n + 2h),$$

t_n is a node point and $\alpha_j, \beta_j, j = 0, 1, 2, \gamma_2$ are parameters to be obtained from the multistep collocation and interpolation techniques. The exact solution $U(t)$ is assumed to exist and unique in $[t_0, t_2]$, and we approximate the exact solution $U(t)$ by seeking the continuous solution $u(t)$ of the form

$$u(t) = \sum_{j=0}^{p+q-1} b_j \phi_j(t), \quad t \in [t_0, t_2] \tag{8}$$

where b_j are unknown coefficients and $\phi_j(t) = t^j, j = 0(1)5$ are the polynomial basis functions of degree 5. The number of interpolation point p and the number of the distinct collocation points q are chosen to satisfy $p = 2$ and $q = 4$. The proposed method is constructed by imposing the following conditions:

$$u_{n+i} = \sum_{j=0}^5 b_j t_{n+i}^j, \quad i = 0, 1 \tag{9}$$

$$f_{n+i} = \sum_{j=0}^5 j b_j t_{n+i}^{j-1}, \quad i = 0(1)2 \tag{10}$$

$$g_{n+i} = \sum_{j=0}^5 j(j-1) b_j t_{n+i}^{j-2}, \quad i = 2 \tag{11}$$

Eqs. (9)–(11) lead to a system of $p + q$ equations which is solved to obtain the coefficients $b_j, j = 0(1)5$. The values of b_j are substituted into (8) to yield the continuous form

$$u(t) = \sum_{j=0}^1 \alpha_j(t) u_{n+j} + h \sum_{j=0}^2 \beta_j(t) f_{n+j} + h^2 \gamma_2(t) g_{n+2} \tag{12}$$

where $\alpha_j(t), j = 0, 1, \beta_j(t), j = 0, 1, 2, \gamma_2$ are continuous coefficients given as

$$\alpha_0(t) = 1 - \frac{120}{23} \left(\frac{t-t_n}{h}\right)^2 + \frac{160}{23} \left(\frac{t-t_n}{h}\right)^3 - \frac{75}{23} \left(\frac{t-t_n}{h}\right)^4 + \frac{12}{23} \left(\frac{t-t_n}{h}\right)^5$$

$$\alpha_1(t) = \frac{120}{23} \left(\frac{t-t_n}{h}\right)^2 - \frac{160}{23} \left(\frac{t-t_n}{h}\right)^3 + \frac{75}{23} \left(\frac{t-t_n}{h}\right)^4 - \frac{12}{23} \left(\frac{t-t_n}{h}\right)^5$$

$$\beta_0(t) = \left(\frac{t-t_n}{h}\right) - \frac{131}{46} \left(\frac{t-t_n}{h}\right)^2 + \frac{265}{92} \left(\frac{t-t_n}{h}\right)^3 - \frac{28}{23} \left(\frac{t-t_n}{h}\right)^4 + \frac{17}{92} \left(\frac{t-t_n}{h}\right)^5$$

$$\beta_1(t) = -\frac{64}{23} \left(\frac{t-t_n}{h}\right)^2 + \frac{116}{23} \left(\frac{t-t_n}{h}\right)^3 - \frac{63}{23} \left(\frac{t-t_n}{h}\right)^4 + \frac{11}{23} \left(\frac{t-t_n}{h}\right)^5$$

$$\beta_2(t) = \frac{19}{46} \left(\frac{t-t_n}{h}\right)^2 - \frac{89}{92} \left(\frac{t-t_n}{h}\right)^3 + \frac{16}{23} \left(\frac{t-t_n}{h}\right)^4 - \frac{13}{92} \left(\frac{t-t_n}{h}\right)^5$$

$$\gamma_2(t) = -\frac{7}{46} \left(\frac{t-t_n}{h}\right)^2 + \frac{17}{46} \left(\frac{t-t_n}{h}\right)^3 - \frac{13}{46} \left(\frac{t-t_n}{h}\right)^4 + \frac{3}{46} \left(\frac{t-t_n}{h}\right)^5$$

The main discrete method (SDBBDF) is generated by evaluating (12) at the point $t = t_{n+2}$ to obtain

$$u_{n+2} = \frac{7}{23} u_n + \frac{16}{23} u_{n+1} + \frac{h}{23} (2f_n + 16f_{n+1} + 12f_{n+2}) - \frac{2}{23} h^2 g_{n+2} \tag{13}$$

Differentiating (12) twice with respect to t , we have

$$u''(t) = \frac{1}{h^2} \left[\sum_{j=0}^1 \overline{\alpha_j(t)} u_{n+j} + h \sum_{j=0}^2 \overline{\beta_j(t)} f_{n+j} + h^2 \overline{\gamma_2(t)} g_{n+2} \right] \quad (14)$$

where

$$\overline{\alpha_0(t)} = -\frac{240}{23} + \frac{960}{23} \left(\frac{t-t_n}{h}\right) - \frac{900}{23} \left(\frac{t-t_n}{h}\right)^2 + \frac{240}{23} \left(\frac{t-t_n}{h}\right)^3$$

$$\overline{\alpha_1(t)} = \frac{240}{23} - \frac{960}{23} \left(\frac{t-t_n}{h}\right) + \frac{900}{23} \left(\frac{t-t_n}{h}\right)^2 - \frac{240}{23} \left(\frac{t-t_n}{h}\right)^3$$

$$\overline{\beta_0(t)} = -\frac{131}{23} + \frac{795}{46} \left(\frac{t-t_n}{h}\right) - \frac{336}{23} \left(\frac{t-t_n}{h}\right)^2 + \frac{85}{23} \left(\frac{t-t_n}{h}\right)^3$$

$$\overline{\beta_1(t)} = -\frac{128}{23} + \frac{696}{23} \left(\frac{t-t_n}{h}\right) - \frac{756}{23} \left(\frac{t-t_n}{h}\right)^2 + \frac{220}{23} \left(\frac{t-t_n}{h}\right)^3$$

$$\overline{\beta_2(t)} = \frac{19}{23} - \frac{267}{46} \left(\frac{t-t_n}{h}\right) + \frac{192}{23} \left(\frac{t-t_n}{h}\right)^2 - \frac{65}{23} \left(\frac{t-t_n}{h}\right)^3$$

$$\overline{\gamma_2(t)} = -\frac{7}{23} + \frac{51}{23} \left(\frac{t-t_n}{h}\right) - \frac{78}{23} \left(\frac{t-t_n}{h}\right)^2 + \frac{30}{23} \left(\frac{t-t_n}{h}\right)^3$$

The additional method is generated by evaluating (14) at the point $t = t_{n+1}$ to obtain

$$h^2 g_{n+1} = \frac{120}{46} u_n - \frac{120}{46} u_{n+1} + \frac{h}{46} (31f_n + 64f_{n+1} + 25f_{n+2}) - \frac{8}{46} h^2 g_{n+2} \quad (15)$$

The methods (13) and (15) are combined to obtain 2-step SDBBDF of order 5 as

$$\left. \begin{aligned} h^2 g_{n+1} &= \frac{120}{46} u_n - \frac{120}{46} u_{n+1} + \frac{h}{46} (31f_n + 64f_{n+1} + 25f_{n+2}) - \frac{8}{46} h^2 g_{n+2} \\ u_{n+2} &= \frac{7}{23} u_n + \frac{16}{23} u_{n+1} + \frac{h}{23} (2f_n + 16f_{n+1} + 12f_{n+2}) - \frac{2}{23} h^2 g_{n+2} \end{aligned} \right\} \quad (16)$$

3. Analysis of SDBBDF

The stability properties, consistency, convergence, local truncation error and order of SDBBDF method are discussed in this section. The method can be represented by a matrix finite difference equation in block form as

$$A_1 U_\omega = A_0 U_{\omega-1} + h(B_1 F_\omega + B_0 F_{\omega-1}) + h^2 D_1 G_\omega \quad (17)$$

where

$$U_\omega = (u_{n+1}, u_{n+2}, u_{n+3}, \dots, u_{n+k-1}, u_{n+k})^T$$

$$U_{\omega-1} = (u_{n-k+1}, u_{n-k+2}, u_{n-k+3}, \dots, u_{n-1}, u_n)^T$$

$$F_\omega = (f_{n+1}, f_{n+2}, f_{n+3}, \dots, f_{n+k-1}, f_{n+k})^T$$

$$F_{\omega-1} = (f_{n-k+1}, f_{n-k+2}, f_{n-k+3}, \dots, f_{n-1}, f_n)^T$$

$$G_\omega = (g_{n+1}, g_{n+2}, g_{n+3}, \dots, g_{n+k-1}, g_{n+k})^T$$

and the matrices A_1, A_0, B_1, B_0 and D_1 are 2 by 2 matrices whose entries are given by the coefficients of Eq. (16) as

$$A_1 = \begin{pmatrix} \frac{120}{46} & 0 \\ -\frac{16}{23} & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & \frac{120}{46} \\ 0 & \frac{7}{23} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{64}{46} & \frac{25}{46} \\ \frac{16}{23} & \frac{12}{23} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & \frac{31}{46} \\ 0 & \frac{2}{23} \end{pmatrix}, \quad D_1 = \begin{pmatrix} -1 & -\frac{8}{46} \\ 0 & -\frac{2}{23} \end{pmatrix}$$

3.1. Local truncation error and order of SDBBDF

The local truncation error associated with 2-step SDBBDF methods can be defined to be the linear difference operator

$$L[u(t_n); h] = \sum_{j=0}^2 [\alpha_j u(t_n + jh) - h\beta_j u'(t_n + jh)] - h^2 \gamma_2 u''(t_n + 2h), \quad (18)$$

Assume that $u(t_n)$ is differentiable as often as needed, then by using Taylor series expansion to expand $u(t_n + jh)$, $u'(t_n + jh)$ and $u''(t_n + 2h)$ in (18) about t_n , we have

$$u(t_n + jh) = \sum_{m=0}^{\infty} \frac{(jh)^m}{m!} u^{(m)}(t_n), \quad u'(t_n + jh) = \sum_{m=0}^{\infty} \frac{(jh)^m}{m!} u^{(m+1)}(t_n)$$

$$u''(t_n + 2h) = \sum_{m=0}^{\infty} \frac{(2h)^m}{m!} u^{(m+2)}(t_n)$$

Substitute $u(t_n + jh)$, $u'(t_n + jh)$ and $u''(t_n + 2h)$ into (18) to obtain

$$L[u(t_n); h] = C_0 u(t_n) + C_1 h u'(t_n) + C_2 h^2 u''(t_n) + \dots + C_m h^m u^{(m)}(t_n) + \dots \quad (19)$$

where $C_m, m = 0, 1, 2, \dots$ are constants given in terms of α_j and β_j

$$\left. \begin{aligned} C_0 &= \sum_{j=0}^2 \alpha_j \\ C_1 &= \sum_{j=0}^2 (j\alpha_j - \beta_j) \\ C_2 &= \frac{1}{2!} \sum_{j=0}^2 j^2 \alpha_j - \sum_{j=0}^2 j\beta_j - \sum_{j=1}^2 \gamma_j \\ &\vdots \\ C_m &= \frac{1}{m!} \left[\sum_{j=0}^2 j^m \alpha_j - m \sum_{j=0}^2 j^{m-1} \beta_j - m(m-1) \sum_{j=1}^2 j^{m-2} \gamma_j \right] \end{aligned} \right\} \quad (20)$$

The block method (16) is said to have a maximal order of accuracy m if

$$C_0 = C_1 = C_2 = \dots = C_m = 0, \quad C_{m+1} \neq 0$$

and if

$$L[u(t_n); h] = C_{m+1} h^{m+1} u^{(m+1)}(t_n)$$

\dot{C}_{m+1} is the error constant and $C_{m+1} h^{m+1} u^{(m+1)}(t_n)$ is the principal local truncation error at the point t_n . Therefore, the values of the error constant calculated for SDBBDF (16) are given as $(\frac{13}{2760}, \frac{1}{1035})^T$ with order $(5, 5)^T$ where T is the transpose.

3.2. Zero-stability

The zero stability of the method in (17) is determined as the limit $h \rightarrow 0$ and the difference system (17) tends to

$$A_1 U_\omega = A_0 U_{\omega-1}$$

whose first characteristic polynomial $\rho(V)$ is given by

$$\rho(V) = \det[VA_1 - A_0] = \frac{60}{23} V(V - 1)$$

Following the definition, the block method (17) are zero stable for $\rho(V) = 0$ and satisfies $|V_j| \leq 1, j = 1, 2$. Thus 2-step Second Derivative Block Backward Differentiation Formulas (SDBBDFs) of order 5 is zero stable.

3.3. Consistency and Convergence

The block method (17) is consistent since it has order $m = 5 > 1$. According to Lambert [6], since the block method is consistent and zero-stable, then the method (17) converges.

3.4. Region of absolute stability

The stability properties of the SDBBDF (17) are determined by applying the derived block formulae to the test equation

$$u' = \lambda u \quad \text{and} \quad u'' = \lambda^2 u \tag{21}$$

Applying (17)–(21) and let $z = \lambda h$, we have

$$A_1 U_\omega = A_0 U_{\omega-1} + zB_1 U_\omega + zB_0 U_{\omega-1} + z^2 D_1 U_\omega$$

which can also be written as

$$U_\omega = R(z)U_{\omega-1}$$

where

$$R(z) = \frac{A_0 + zB_0}{A_1 - zB_1 - z^2 D_1}$$

$R(z)$ is an amplification matrix that has eigenvalues of the form $(\mu_1, \mu_2) = (0, \mu_2)$ where the dominant eigenvalue μ_2 is a rational function dependent on z given by

$$\mu_2(z) = \frac{2.6087 + 2.08696z + 0.652174z^2 + 0.0869565z^3}{2.6087 - 3.13043z + 1.69565z^2 - 0.521739z^3 + 0.0869565z^4}$$

$\mu_2(z)$ is the Stability Function of SDBBDF (17). The region of absolute stability (RAS) of the 2-step SDBBDF is plotted using the boundary locus techniques. RAS plots the real (x -axis) against the imaginary (y -axis).

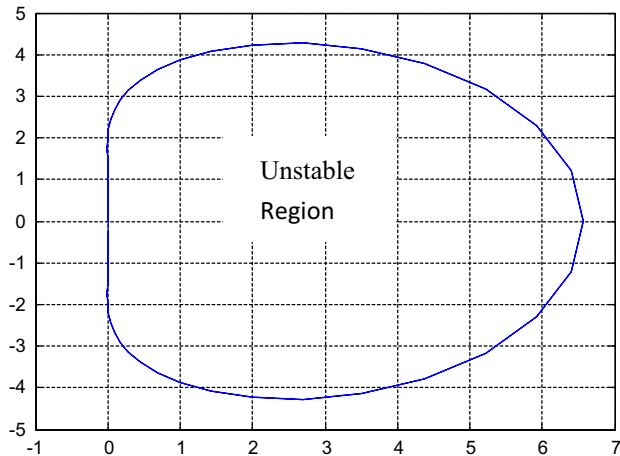


Figure 1 RAS of 2-step SDBBDF.

In Fig. 1, the unstable region is the interior of the curve while outside the curve is the stable region which corresponds to the 2-step SDBBDF (17). Clearly, it is obvious that the method is L-stable since the stability region contains the entire left half complex plane (A-stable) and in addition $\lim_{z \rightarrow \infty} \mu_2(z) = 0$.

4. Implementation of the method

The resulting system of ODEs subject to initial conditions is solved on the partition $\pi_N : \{t_0 < t_1 < \dots < t_N, t_n = t_0 + nh\}$ where $h = \Delta t = \frac{t_N - t_0}{N}$ is a constant step size of the partition and $n = 1, 2, \dots, N, N$ is a positive integer and n is the grid index.

Step 1: Use the block method (16) to solve the resulting ODEs on rectangles

$$[t_0, t_2] \times [0, 1], \quad [t_2, t_4] \times [0, 1], \dots, \quad [t_{N-2}, t_N] \times [0, 1].$$

Step 2: Let $U_{m,\mu} = (u_{m,n+1}, u_{m,n+2})^T, m = 1, 2, \dots, M$ and $u_{m,n} \cong u(x_m, t_n)$ then for $m = 1, 2, \dots, M, n = 0$ and $\mu = 1$, the approximations $(u_{m,1}, u_{m,2})^T$ are simultaneously obtained over the sub-interval $[t_0, t_2]$, as t_0 is known from the problem.

Step 3: Step 2 is repeated for $m = 1, 2, \dots, M, n = 2$ and $\mu = 2$, the approximations $(u_{m,3}, u_{m,4})^T$ are simultaneously obtained over the sub-interval $[t_2, t_4]$, as t_2 is known from the previous block. The process is continued until the approximates $(u_{m,N-1}, u_{m,N})^T$ are obtained simultaneously over the sub-interval $[t_{N-2}, t_N]$.

5. Numerical results

This section deals with some numerical examples executed in our written program in MAPLE 17 to show the efficiency of the derived block method on parabolic equations.

Example 1. Consider the PDE (1) subject to the initial condition (2) and integral boundary conditions (3) and (4), [see Dehghan [9]] with the following

$$\begin{aligned} f(x) &= x^2, \\ Q(x, t) &= \frac{-2(x^2 + t + 1)}{(t + 1)^3}, \\ s_1(t) &= -\frac{1}{4(t + 1)^2}, \\ s_2(t) &= \frac{3}{4(t + 1)^2}, \\ r_1(x, t) &= x, \\ r_2(x, t) &= x, \end{aligned}$$

The exact solution is $U(x, t) = \frac{x^2}{(t+1)^2}$. Results obtained in [9,16] were reproduced and compared with that obtained with the newly derived SDBBDF in Tables 1 and 2. For this problem, the SDBBDF performed better than the existing methods in [9,16].

Table 1 Relative errors of numerical values of $u(0.6, 1.0)$ for Example 1.

Δt	BTCS [9]	Crandall [9]	FTCS [9]	Dufort–Frankel [9]	Li and Wu [16]	SDBBDF
0.05	7.3E–02	3.8E–03	7.5E–02	7.8E–02	1.6E–03	5.4E–05

BTCS – Backward Time Centered Space, FTCS – Forward Time Centered Space.

Table 2 Numerical results for Example 1 at $t = 1$.

x	Exact ($U(x, t)$)	SDBBDF ($u(x, t)$)	Error ($U(x, t) - u(x, t)$)
0.1	0.0025	0.0024755	2.444E–05
0.2	0.01	0.0099652	3.473E–05
0.3	0.0225	0.0224587	4.123E–05
0.4	0.04	0.0399542	4.575E–05
0.5	0.0625	0.0624506	4.938E–05
0.6	0.09	0.0899456	5.434E–05
0.7	0.1225	0.12245160	4.839E–05
0.8	0.16	0.16006866	6.867E–05
0.9	0.2025	0.20313845	6.385E–04

Example 2. Next, consider the PDE of the form (1) subject to the initial condition (2) and integral boundary conditions (3) and (4) with the following

$$f(x) = 1 + \cos(x),$$

$$Q(x, t) = 0,$$

$$s_1(t) = \frac{1}{2} + e^{-t} - t - e^{-t}(-1 + \cos(1) + \sin(1) + t \sin(1)),$$

$$s_2(t) = 1 + e^{-t} \cos(1) - \frac{t}{2e} (2(-1 + e) + e^{-t}(e - \cos(1) + \sin(1))),$$

$$r_1(x, t) = x + t,$$

$$r_2(x, t) = te^{-x},$$

which has exact solution $U(x, t) = 1 + e^{-t} \cos(x)$.

In Table 3 the numerical results of example 2 at $t = 1$ using $\Delta t = 10^{-2}$ and 10^{-3} solved with SDBBDF are displayed.

Table 3 Numerical results for Example 2 at $t = 1$ using SDBBDF.

Δt	x								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
10^{-2}	8.82E–04	1.87E–05	3.15E–05	3.09E–05	3.86E–05	3.41E–05	3.12E–05	1.07E–05	5.45E–04
10^{-3}	4.41E–06	1.06E–06	4.12E–07	5.54E–07	5.37E–07	5E–07	3.75E–07	4.67E–07	2.43E–06

Table 4 A comparison of errors of methods for Example 3 at $t = 1, \mu = 1, \Delta x = 0.1, \Delta t = 0.1$.

w	SDBBDF	BHSDA	Crank–Nicolson	Cash (AB)	Cash (ABC)
1	2.69E–06	2.64E–06	6.20E–05	3.7E–05	1.5E–06
2	1.35E–06	1.32E–06	3.83E–05	1.8E–05	7.4E–06
3	1.35E–06	1.32E–06	9.30E–03	1.9E–05	7.4E–06
5	1.35E–06	1.32E–06	1.80E–01	1.8E–05	7.4E–06
10	1.35E–06	1.32E–06	6.10E–01	1.8E–05	7.4E–06

Example 3. Lastly, we consider the stiff type parabolic partial differential equation which has also been solved by cash [8] and Ngwane and Jator [5] of the form

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(1, t) = 0$$

$$u(x, 0) = \sin(\pi x) + \sin(w\pi x), \quad w \gg 1$$

The exact solution $u(x, t) = e^{-\pi^2 \mu t} \sin(\pi x) + e^{-w^2 \pi^2 \mu t} \sin(w\pi x)$.

Cash [8] notes that as w increases, equations of the type given in example 3 exhibit characteristics similar to model stiff equations. Hence the method such as the Crank–Nicolson method, which is neither L_0 -stable nor L -stable, is expected to perform poorly. The SDBBDF is L -stable and compares well with BHSDA [5] but performs better than the L_0 -stable methods of Cash [8]. In Table 4, the numerical results for $\mu = 1$ and a range of w are displayed.

6. Conclusion

An L -stable Second Derivative Block Backward Differentiation Formula (SDBBDF) has been employed successfully for the numerical solution of parabolic partial differential equations. Applying the classical method of lines, the parabolic equation subject to boundary and initial conditions is converted into a system of Ordinary Differential Equations (ODE) subject to initial conditions. Then the resulting ODE is solved using SDBBDF. The numerical results show that SDBBDF is accurate and reliable for the class of problems considered.

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